

The dependence of the atomic energy levels on a superstrong magnetic field with account of a finite nucleus radius and mass

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Abstract

The influence of the finiteness of the proton radius and mass on the energies of a hydrogen atom and hydrogen-like ions in a superstrong magnetic field is studied. The finiteness of the nucleus size pushes the ground energy level up leading to a nontrivial dependence of the value of critical nucleus charge on the external magnetic field.

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I. INTRODUCTION

At magnetic fields $B \geq (3\pi/e^2)B_0 \equiv 3\pi m_e^2/e^3 \approx 6 \cdot 10^{16}$ G¹ the Coulomb potential of the nucleus becomes screened due to large radiative corrections [1] ($B_0 \equiv m_e^2/e = 4.4 \cdot 10^{13}$ G = $4.4 \cdot 10^9$ T). This leads in particular to the freezing of the ground state energy of the hydrogen atom at the value $E_0 = -1.7$ keV [1, 2]. This statement is correct up to the values of the magnetic field at which the Landau radius $a_H \equiv 1/\sqrt{eB}$ becomes close to the radius of the nucleus, $a_H \approx R$. For hydrogen this happens at $B \approx 10^5 B_0 \approx 10^{19}$ G, where the value of the proton charge radius $R = 0.877$ fm (see [3]) was used for numerical estimate. The approximation of a pointlike proton is not valid for $B \gtrsim 10^{19}$ G, and in Sec. II and III we will find to what changes of atomic energies the finiteness of the proton size leads.

With the growth of the nuclei charge Z the energy of the ground electron level diminishes and in the absence of magnetic field a nucleus with $Z = 172$ is critical: the ground level sinks into the lower continuum. As soon as the charge of the ion reaches $Z_{cr} = 172$ two e^+e^- -pairs are produced from the vacuum. Electrons with opposite spins occupy the ground energy level while positrons are emitted to infinity [4]. In an external magnetic field the value of Z_{cr} diminishes [5, 6]. According to [5] at $B \approx 10^2 B_0$ uranium becomes critical, $Z_{cr} = 92$, while at $B = 10^4 B_0$ even comparatively light nuclei are critical, $Z_{cr} \approx 40$. These results were obtained without taking the screening of the Coulomb potential into account. It was accounted for in [7] where it was found that because of screening larger magnetic fields are needed for a particular nucleus to become critical. Even more: according to [7] nuclei with $Z < 50$ do not reach criticality because of screening. The approximation of pointlike nuclei (which is valid when the Landau radius is considerably larger than the size of the nucleus) was used in [7]. A nucleus with $Z \approx 56$ becomes critical when the magnetic field is so large that the Landau radius equals the size of the nucleus. It was noted in [7] that the diminishing of the Coulomb potential due to the finite radius of the nucleus should push the value of the ground energy level up, and it was expected that this phenomenon will prevent ions with $Z = 50, 51$ from reaching criticality. In Sec. IV we present quantitative consideration on the influence of the finiteness of nucleus size on the dependence of Z_{cr} on the value of the external magnetic field B . This dependence turns out to be not so simple. In particular, only nuclei with $Z > 59$ reach criticality in a superstrong magnetic fields.

¹ We use the system of units in which $\hbar = c = 1$, $\alpha = e^2 = 1/137.03599 \dots$

When B further grows even nuclei with $Z > 59$ become noncritical (see Fig. 3).

In [1, 2] and [7] the atomic nucleus was considered as an infinitely heavy source of the Coulomb field. Because of the finiteness of the mass of the nucleus its motion in the magnetic field should be taken into account and one should consider the two body (electron and nucleus) problem in the presence of a homogeneous magnetic field. This consideration was made in [8–10]. According to the results obtained in [10] hydrogen atomic levels get increased by $e|m|B/m_p$, where m is the projection of the relative angular momentum on the direction of the external magnetic field and m_p is the proton mass. The corresponding formulae and numerical estimates are presented in Sec. V.

The finiteness of the nucleus mass also leads to a nonzero hyperfine interaction between the spins of the proton and the electron. Its importance in the case of a superstrong B was stressed in [11] and we present our comments in Sec. VI.

II. ELECTRIC POTENTIAL OF THE NUCLEUS

The finite size of the nucleus makes the Coulomb potential less singular at small distances, pushing up the electron levels. However the shape of potential at distances much smaller than the Bohr radius a_B is not very important for the values of the electron energies. The effect is the largest for S levels, where the relative shift of energies goes like $(R/a_B)^2 \approx 10^{-10}$ where R is nucleus radius (in case of muonic atoms this shift is more important since it is enhanced by $(m_\mu/m_e)^2 \sim 10^5$).

Strong magnetic fields make the Coulomb problem essentially one-dimensional. And in one space dimension a $1/|z|$ potential leads to a spectrum unbounded from below: the ground state energy equals minus infinity. The divergence of the potential at $z \rightarrow 0$ is regulated by the Landau radius a_H : $|V(z)| \lesssim e^2/a_H$. It follows from this consideration that the behaviour of the potential at small distances determines the energy of the ground state. In this section we will find how accounting for the finite size of the nucleus modifies its electric potential.

An analytic formula for the Coulomb potential of a pointlike charge along the z axis screened by a magnetic field was derived in [2]:

$$\Phi(0, z) = \frac{e}{|z|} \left(1 - e^{-|z|\sqrt{6m_e^2 + (2e^3 B/\pi)}} + e^{-\mu|z|} \right), \quad (1)$$

where z is the coordinate along the magnetic field, $\mu \equiv \sqrt{6m_e^2 + (2e^3 B/\pi)}$, and the charge

is located at the point $z = 0$. The sum of the first two terms, $\left(1 - e^{-|z|\sqrt{6m_e^2}}\right)/|z|$, does not vary at distances $z \ll 1/m_e$, and the smearing of the pointlike charge along the z axis within the domain with size $R \ll 1/m_e$ does not affect it. At the same time the last term, $e^{-\mu|z|}/|z|$, becomes very sensitive to the charge distribution for $B \gtrsim 1/(e^3 R^2)$, $\mu R \gtrsim 1$.

The potential in the plane transverse to the magnetic field ($z = 0$) was found in [7] for $\mu \gg m_e$, $B \gg m_e^2/e^3$:

$$\Phi(\rho, 0) = \frac{e}{\rho} \left(e^{-\mu\rho} + \frac{\sqrt{6m_e^2}}{\mu} \right), \quad (2)$$

where $\vec{\rho}$ is the coordinate in the transverse plane. The potential has a Yukawa behaviour both in the direction transverse to the magnetic field and along the z axis at distances $\rho, z \lesssim l_0 \equiv \frac{1}{\mu} \ln \frac{\mu}{\sqrt{6m_e^2}}$. At these distances the potential is sensitive to the charge distribution. The important question is whether the long range part of the potential along the z axis, $\left(1 - e^{-|z|\sqrt{6m_e^2}}\right)/|z|$, is also affected by the smearing of the pointlike charge in the transverse plane.

At $z \gg 1/m_e$ the potential $\Phi(\rho, z)$ has the following behaviour [7]:

$$\Phi(\rho, z) = \frac{e}{\sqrt{z^2 + \left(1 + \frac{e^3 B}{3\pi m_e^2}\right)\rho^2}}. \quad (3)$$

The main contribution from the terms $\left(1 - e^{-|z|\sqrt{6m_e^2}}\right)/|z|$ to the value of the electron energy comes from large distances, $1/m_e \ll z \ll 1/(e^2 m_e)$. To significantly change this contribution the potential (3) should noticeably differ from the Coulomb potential at distances $z \sim 1/(m_e e)$:

$$\frac{e^3 B}{3\pi m_e^2} R^2 \gtrsim \frac{1}{(em_e)^2} \Rightarrow B \gtrsim \frac{3\pi}{e^5 R^2} \approx 4 \cdot 10^{10} B_0. \quad (4)$$

We are not going to consider here such strong fields; so this effect is neglected in what follows.

Thus, there are two parts in the potential: the first one, $\left(1 - e^{-|z|\sqrt{6m_e^2}}\right)/|z|$, does not depend on the charge distribution inside the nucleus and the second one, originating from $e^{-\mu r}/r$ (where $r = \sqrt{\rho^2 + z^2}$), is determined by the charge distribution.

Another issue is the modification of the proton shape in a superstrong magnetic fields which would lead to the variation of atomic energies. As soon as the Landau radius of the electron becomes close to the proton radius the same happens with the Landau radius

of the proton. When the magnetic field further grows one could expect that the size of the proton in the direction perpendicular to the magnetic field shrinks. But the proton is not an elementary particle and for $R \gg a_H$ the valence and sea quarks will be oscillating inside domains with size of the order of a_H which are not necessarily situated at $\rho = 0$. The distribution of these rotating quarks inside the nucleus will be defined by strong interactions, so there is no reason to think that the nucleus in the transverse plane is squeezed down to the size of a_H .² What is really happening to the proton shape in such a strong magnetic field is a subject for a separate study, while here we will neglect the possible shrinking of the nucleus in the magnetic field.

With the account of the specific features discussed above the potential along the z axis at $\rho \lesssim a_H$ has the following form:

$$\Phi(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + h(R)e^{-\mu r} \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + h(r)e^{-\mu R} \right), & r < R, \end{cases} \quad (5)$$

where $h(r)$ is determined by the charge distribution inside the nucleus.

Since the charge distribution inside the nucleus in such strong magnetic field is not known, we will consider three following cases:

1. “Simple cut” — the potential outside the nucleus is equal to that of a pointlike charge and inside the nucleus it is constant and equals the value of the potential at the surface.

This case corresponds to $h(r) = 1$:

$$\Phi^{(1)}(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + e^{-\mu r} \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \right), & r < R. \end{cases} \quad (6)$$

2. Homogeneously charged sphere — taking into account that the last term $\varphi = eh(r)e^{-\mu R}/R$ in the expression (5) satisfies the Yukawa equation $\Delta\varphi - \mu^2\varphi = -4\pi Q(r)$ we obtain:

$$\Phi^{(2)}(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + e^{-\mu r} \cdot \frac{1}{2\mu R} (e^{\mu R} - e^{-\mu R}) \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \cdot \frac{1}{2\mu r} (e^{\mu r} - e^{-\mu r}) \right), & r < R. \end{cases} \quad (7)$$

² Nuclear core should prevent heavy ions from shrinking in the direction transverse to the magnetic field.

3. Homogeneously charged ball — the potential can be easily found from the formula for a charged sphere:

$$\Phi^{(3)}(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + e^{-\mu r} \cdot \frac{3}{2(\mu R)^3} \left(e^{\mu R}(\mu R - 1) + e^{-\mu R}(\mu R + 1) \right) \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \cdot \frac{3}{(\mu R)^2} \left(e^{\mu R} - \frac{(\mu R + 1)}{2\mu r} (e^{\mu r} - e^{-\mu r}) \right) \right), & r < R. \end{cases} \quad (8)$$

III. THE FINITE SIZE OF THE PROTON AND THE HYDROGEN ATOMIC LEVELS

The following equation for the hydrogen atomic energies on which the lowest Landau level (LLL) with $m = 0$ splits was obtained in [12] by solving the Schrödinger equation (and it was checked in [7] that the relativistic corrections can be neglected as far as the binding energy is much smaller than the electron mass):

$$2 \ln \frac{z_0}{a_B} + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + 4\gamma + 2 \ln 2 = 2 \int_0^{z_0} dz \int \frac{|R_{00}(\rho)|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \equiv I, \quad (9)$$

where the energies of the atomic states are determined by λ , $E \equiv -(m_e e^4/2) \lambda^2$, $a_B \equiv 1/(m_e e^2)$ is the Bohr radius, ψ is the logarithmic derivative of the gamma function, and $\gamma = 0.5772\dots$ is the Euler's constant. $R_{00}(\rho) = e^{-\rho^2/4a_H^2}/\sqrt{2\pi a_H^2}$ is the wave function which corresponds to the transverse motion of the electron occupying the ground ($n_\rho = m = 0$) Landau level. The dependence on the matching point z_0 cancels in (9) for $a_H \ll z_0 \ll 1/(m_e e^2)$.

To take screening into account the factor $1/\sqrt{\rho^2 + z^2}$ in the right hand side of (9) should be substituted by:

$$\frac{1}{\sqrt{\rho^2 + z^2}} \rightarrow \frac{1}{\sqrt{\rho^2 + z^2}} \left(1 - e^{-\sqrt{\rho^2 + z^2}\sqrt{6m_e^2}} + e^{-\sqrt{\rho^2 + z^2}\sqrt{6m_e^2 + 2e^3 B/\pi}} \right), \quad (10)$$

which leads to the freezing of the values of atomic energies at $B \gg m_e^2/e^3$ [1, 2]. To take the finite proton size into account instead of (10) one should make the following substitution in (9):

$$\frac{1}{\sqrt{\rho^2 + z^2}} \rightarrow \Phi(\rho, z)/e, \quad (11)$$

where $\Phi(\rho, z)$ is given by (5). For the right hand side of (9) we get:

$$I = I_1 + I_2 + I_3 \equiv 2 \int_0^R dz \int_0^{\sqrt{R^2 - z^2}} 2\pi \rho d\rho |R_{00}(\rho)|^2 \frac{1}{R} \left[1 - e^{-R\sqrt{6m_e^2}} + h(\sqrt{\rho^2 + z^2})e^{-\mu R} \right] +$$

$$\begin{aligned}
& + 2 \int_0^R dz \int_{\sqrt{R^2 - z^2}}^{\infty} 2\pi \rho d\rho |R_{00}(\rho)|^2 \frac{1}{\sqrt{\rho^2 + z^2}} \left[1 - e^{-\sqrt{\rho^2 + z^2} \sqrt{6m_e^2}} + h(R) e^{-\mu \sqrt{\rho^2 + z^2}} \right] + \\
& + 2 \int_R^{z_0} dz \int_0^{\infty} 2\pi \rho d\rho |R_{00}(\rho)|^2 \frac{1}{\sqrt{\rho^2 + z^2}} \left[1 - e^{-\sqrt{\rho^2 + z^2} \sqrt{6m_e^2}} + h(R) e^{-\mu \sqrt{\rho^2 + z^2}} \right]. \quad (12)
\end{aligned}$$

The sum $I_1 + I_2$ for $a_H \approx R$ is of order 1 and can be safely neglected. For $a_H \ll R$ we have:

$$I_1 + I_2 \approx 2R \cdot \frac{1}{R} \left(1 - e^{-R\sqrt{6m_e^2}} \right) + \frac{2e^{-\mu R}}{R} \int_0^R h(z) dz \equiv 2 \left(1 - e^{-R\sqrt{6m_e^2}} + \langle h \rangle e^{-\mu R} \right). \quad (13)$$

We see that for three examples considered in Sec. II the sum $I_1 + I_2$ rapidly diminishes when B grows, so we can safely neglect these terms.³

For I_3 we have:

$$\begin{aligned}
I_3 & \approx 2 \int_{\sqrt{R^2 + a_H^2}}^{z_0} \frac{dz}{z} \left[1 - e^{-z\sqrt{6m_e^2}} + h(R) e^{-\mu z} \right] \approx \\
& \approx 2 \left[\ln \frac{z_0}{\sqrt{R^2 + a_H^2}} - E_1 \left(\sqrt{R^2 + a_H^2} \sqrt{6m_e^2} \right) + h(R) E_1 \left(\mu \sqrt{R^2 + a_H^2} \right) \right], \quad (14)
\end{aligned}$$

where

$$\begin{aligned}
E_1(x) & \equiv \int_x^{\infty} \frac{e^{-t}}{t} dt, \\
E_1(x)|_{x \ll 1} & = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}, \\
E_1(x)|_{x \gg 1} & = \frac{e^{-x}}{x} \left(1 - \frac{1}{x} + \frac{1 \cdot 2}{x^2} - \frac{1 \cdot 2 \cdot 3}{x^3} + \dots \right). \quad (15)
\end{aligned}$$

Substituting (14) in (9) we get an equation which determines the values of λ and the corresponding values of the energies on which the Landau level with $n_\rho = m = 0$ splits by the screened Coulomb potential:

$$\begin{aligned}
& \ln \frac{z_0}{a_B} + \frac{\lambda}{2} + \ln \lambda + \psi \left(1 - \frac{1}{\lambda} \right) + 2\gamma + \ln 2 = \\
& \ln \frac{z_0}{\sqrt{R^2 + a_H^2}} - E_1 \left(\sqrt{R^2 + a_H^2} \sqrt{6m_e^2} \right) + h(R) E_1 \left(\mu \sqrt{R^2 + a_H^2} \right). \quad (16)
\end{aligned}$$

³ Let us note that the term $\langle h \rangle e^{-\mu R}$ is not necessarily decreasing with the magnetic field because $h(r)$ could correspond to any charge distribution inside the nucleus including a pointlike distribution. In case of a pointlike distribution this term would prevent the ground energy level from going up.

The dependence on the matching point z_0 cancels and finally we obtain the equation which determines the values of the freezing atomic energies with the account of the finite proton size:

$$\frac{\lambda}{2} + \ln \lambda + \psi \left(1 - \frac{1}{\lambda} \right) + 2\gamma + \ln 2 = \ln \frac{a_B}{\sqrt{R^2 + a_H^2}} - E_1 \left(\sqrt{R^2 + a_H^2} \sqrt{6m_e^2} \right) + h(R) E_1 \left(\mu \sqrt{R^2 + a_H^2} \right). \quad (17)$$

In the limit $B \gg 1/(e^3 R^2)$ the right hand side of (17) does not depend on R and we obtain:

$$I_3|_{B \gg 1/(e^3 R^2)} = 2 \left(\ln z_0 \sqrt{6m_e^2} + \gamma \right), \quad (18)$$

$$\lambda^{\text{lim}} + 2 \ln \lambda^{\text{lim}} + 2\psi \left(1 - \frac{1}{\lambda^{\text{lim}}} \right) + 2\gamma + 2 \ln 2 = \ln \left(\frac{6}{e^4} \right). \quad (19)$$

There are two ways to satisfy the equation (19) which has the large logarithm in the right hand side. The first one is to take a large λ^{lim} which will correspond to the ground energy level. The second one is to choose λ^{lim} close to the poles of $\psi(1 - \frac{1}{\lambda^{\text{lim}}})$. The logarithmic derivative of the gamma function $\psi(x)$ has poles at $x = 0, -1, -2, \dots$ which defines a series of $\lambda^{\text{lim}} \approx 1/n$, $n = 1, 2, \dots$. This tower of λ^{lim} corresponds to the well known Balmer series of hydrogen atomic energies $E_n^{\text{lim}} \approx -(m_e e^4)/(2n^2)$.

For the ground level from (19) we obtain $\lambda^{\text{lim}} = 6.9$ instead of the value for a pointlike proton $\lambda^{\text{lim}} = 11.2$ obtained in [2]. Let us note that for $B \gg 1/(e^3 R^2)$ all our approximations have a very good accuracy, so this result should have a good accuracy as well. To check it we solved the Schrödinger equation numerically. Since the adiabatic approximation is applicable ($a_H \ll a_B$, $B \gg m_e^2 e^3$) one has to solve the one dimensional Schrödinger equation with an effective potential $\bar{V}(z)$:

$$\frac{d^2 \chi}{dz^2} + 2m_e(E - \bar{V})\chi = 0, \quad (20)$$

$$E \equiv -\frac{m_e e^4}{2} \lambda^2, \quad \bar{V}(z) \equiv -\frac{e}{a_H^2} \int_0^\infty \Phi(\rho, z) \exp \left(-\frac{\rho^2}{2a_H^2} \right) \rho d\rho.$$

In order to numerically solve equation (20) an analytical formula for the averaged potential energy $\bar{V}(z)$ is needed. For $B \gg 1/(eR^2)$, $a_H \ll R$, the potentials (6)–(8) do not vary with ρ for $\rho \lesssim a_H$, so the averaging over the transverse direction does not change them, i.e. $\bar{V}^{(i)}(z) \approx -e\Phi^{(i)}(0, z)$, $i = 1, 2, 3$. For $B \ll 1/(eR^2)$, $a_H \gg R$, the modification of

the potential at distances $r < R$ does not affect the electron motion while the averaging is very important since it removes $1/r$ singularity of $\Phi(\rho, z)$. So we need an analytical formula for the averaged potential energy for intermediate fields, $B \sim 1/(eR^2)$ at which $a_H \sim R$. However at distances $r \sim R \sim a_H$ the screening does not occur (because $1/\mu \approx 10a_H$) so for $B \sim 1/(eR^2)$ one should average the non-screened potential at these distances.

Without taking screening into account the potential energy of the electron in the external electric potential of a homogeneously charged sphere has the following form:

$$\Phi^{(0)}(\rho, z) = \begin{cases} \frac{e}{\sqrt{\rho^2 + z^2}}, & \sqrt{\rho^2 + z^2} \geq R, \\ \frac{e}{R}, & \sqrt{\rho^2 + z^2} < R. \end{cases} \quad (21)$$

The formula for the corresponding averaged potential energy $\bar{V}^{(0)}(z)$ looks like:

$$\bar{V}^{(0)}(z) = \begin{cases} -e^2 \left(\frac{1}{R} \left(1 - e^{(z^2 - R^2)/2a_H^2} \right) + \frac{1}{a_H} \sqrt{\frac{\pi}{2}} e^{z^2/2a_H^2} \operatorname{erfc} \left(\frac{R}{a_H \sqrt{2}} \right) \right), & |z| < R, \\ -e^2 \frac{1}{a_H} \sqrt{\frac{\pi}{2}} e^{z^2/2a_H^2} \operatorname{erfc} \left(\frac{|z|}{a_H \sqrt{2}} \right), & |z| \geq R, \end{cases} \quad (22)$$

where $\operatorname{erfc}(x)$ is the complementary error function:

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy. \quad (23)$$

The extension of the formula (22) to the entire range of distances and magnetic fields for the potential energies $\bar{V}^{(1)}(z)$ and $\bar{V}^{(2)}(z)$ which correspond to the potentials $\Phi^{(1)}(\rho, z)$ and $\Phi^{(2)}(\rho, z)$ is given by ($i = 1, 2$):

$$\bar{V}^{(i)}(z) = \begin{cases} -e^2 S_1^{(i)} \left(\frac{1}{R} \left(1 - e^{(z^2 - R^2)/2a_H^2} \right) + \frac{1}{a_H} \sqrt{\frac{\pi}{2}} e^{z^2/2a_H^2} \operatorname{erfc} \left(\frac{R}{a_H \sqrt{2}} \right) \right), & |z| < R, \\ -e^2 S_2^{(i)} \frac{1}{a_H} \sqrt{\frac{\pi}{2}} e^{z^2/2a_H^2} \operatorname{erfc} \left(\frac{|z|}{a_H \sqrt{2}} \right), & |z| \geq R; \end{cases} \quad (24)$$

$$S_1^{(1)} = \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \right), \quad S_2^{(1)} = \left(1 - e^{-|z|\sqrt{6m_e^2}} + e^{-\mu|z|} \right); \quad (25)$$

$$S_1^{(2)} = \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \cdot \frac{1}{2\mu|z|} \left(e^{\mu|z|} - e^{-\mu|z|} \right) \right), \quad (26)$$

$$S_2^{(2)} = \left(1 - e^{-|z|\sqrt{6m_e^2}} + e^{-\mu|z|} \cdot \frac{1}{2\mu R} \left(e^{\mu R} - e^{-\mu R} \right) \right). \quad (27)$$

The formula (24) has the correct behaviour both for $B \ll 1/(e^3 R^2)$, $a_H/e \gg R$, (because the screening factors $S_1^{(i)} \approx 1$ and $S_2^{(i)} \approx 1$ for $|z| < R$) and for $B \gg 1/(eR^2)$, $a_H \ll R$, (because averaging does not change $\Phi^{(0)}(\rho, z)$: $\bar{V}^{(0)}(z)|_{B \gg 1/(eR^2)} \approx -e\Phi^{(0)}(0, z)$). Since these ranges of magnetic fields overlap the formula (24) is correct for all B .

In Fig. 1 the behaviour of λ^{gr} (which corresponds to the ground level) according to the analytical expression (17) for $h(r) = 1$ ("Simple cut") and according to the results of the

numerical solution of Eq. (20) with $\bar{V}^{(1)}(z)$ and $\bar{V}^{(2)}(z)$ are shown. We see how λ^{gr} is going down (or the ground level is going up) when the Landau radius a_H becomes of the order of the proton radius ($B \sim 1/(eR^2) \approx 2 \cdot 10^5 B_0$). The raising stops and the energy freezes at $B \sim 1/(e^3 R^2) \approx 3 \cdot 10^7 B_0$. According to Fig. 1 $\bar{V}^{(1)}(z)$ and $\bar{V}^{(2)}(z)$ lead to practically the same dependence of λ^{gr} on B so we use the potential $\bar{V}^{(1)}(z)$ in what follows.⁴

To present a qualitative explanation of the phenomenon of the rising of the ground energy level let us put $z_0 \approx a_B$ in Eq. (9). In case of a pointlike charge the main contributions to I (the right hand side of (9)) for $B \gg m_e^2/e^3$ come from integrating over $a_H < |z| < 1/\mu$ and

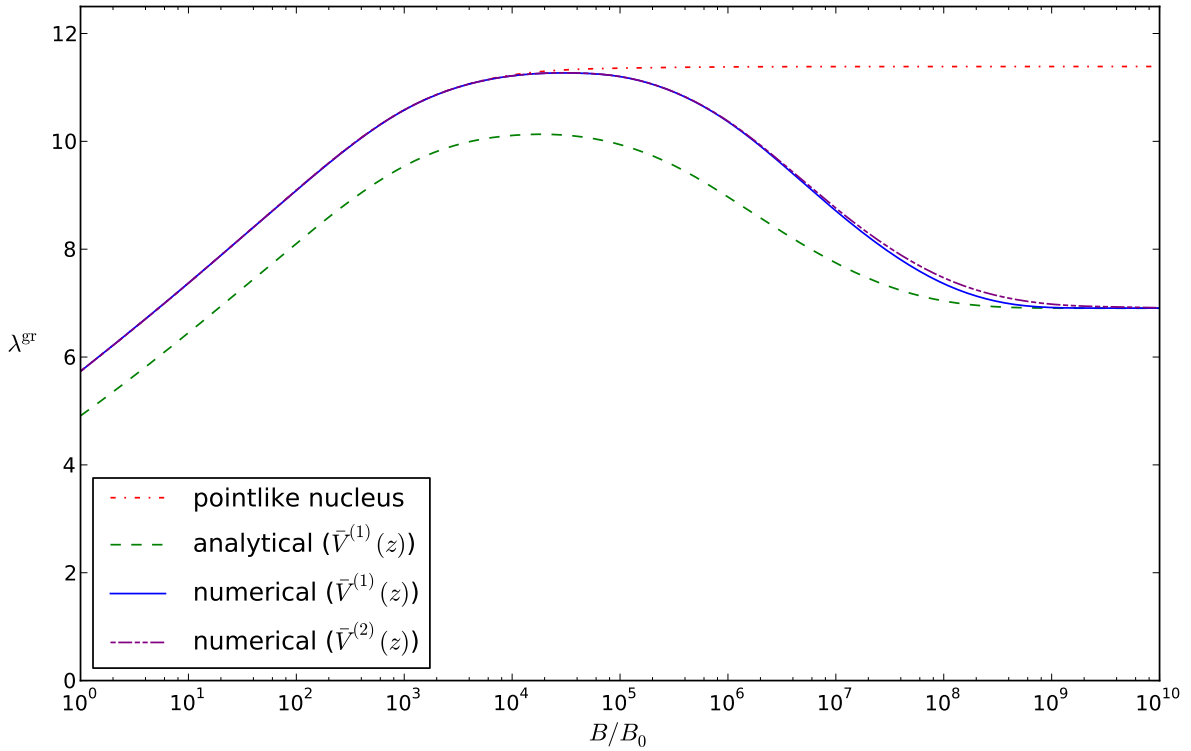


FIG. 1. The dependence of λ^{gr} on the magnetic field; the ground state energy equals $E_0 \equiv -(m_e e^4/2) (\lambda^{\text{gr}})^2$. The dot-dashed (red) line corresponds to the pointlike nucleus, the dashed (green) line — to the analytical formula (17), the solid (blue) and the dashed line with two dots (purple) — to the numerical solutions of (20) with $\bar{V}^{(1)}(z)$ and $\bar{V}^{(2)}(z)$ correspondingly.

⁴ The potential $\Phi^{(3)}(\rho, z)$ varies inside the nucleus even in the non-screened case, that is why we do not have an analytical formula for the averaged potential energy. Nevertheless we made an estimate for $\bar{V}^{(3)}(z)$ and found the numerical results for λ^{gr} which appeared to be rather close to the results obtained with $\bar{V}^{(1)}(z)$ and $\bar{V}^{(2)}(z)$.

$1/m_e < |z| < z_0 \approx a_B$ where the potential has the Coulomb ($1/|z|$) behaviour:

$$I|_{eB < 1/R^2} \approx \ln \frac{1}{a_H^2 \mu^2} + \ln (m_e^2 a_B^2) \approx \ln \frac{1}{e^2} + \ln \frac{1}{e^4} = \ln \frac{1}{e^6}. \quad (28)$$

When the magnetic field grows the Landau radius a_H approaches the proton radius R and the first logarithm in (28) should be substituted by $\ln(1/(R\mu)^2)$ since at $|z| < R$ the potential does not have a $1/|z|$ behaviour. When B further grows and $e^3 B$ approaches $1/R^2$ the first logarithm in (28) goes away:

$$I|_{e^3 B \approx \mu^2 > 1/R^2} \approx \ln (m_e^2 a_B^2) = \ln \frac{1}{e^4}. \quad (29)$$

The decreasing of I corresponds to the diminishing of λ and, therefore, the ground energy level goes up.

IV. CRITICAL NUCLEUS CHARGE

It is well known that the Dirac equation in a pointlike Coulomb potential is not self-consistent for an electric charge $Z > 137$. If the finite size of the nucleus is taken into account then the Dirac equation becomes self-consistent. For $Z \approx 172$ the ground energy level reaches the lower continuum, $\varepsilon = -m_e$, and two electron-positron pairs are created from the vacuum [4]. The electrons occupy the ground energy level while the positrons become free (two pairs are created due to the spin degeneracy of the ground energy level). This is known as the critical nucleus charge phenomenon.

In the presence of a magnetic field the value of the critical nucleus charge diminishes [5]. The atomic energies were found by solving the Dirac equation. In [5] it was transformed into a set of two one-dimensional differential equations of first order which can be done when the adiabatic approximation is valid ($B \gg B_0 (Ze^2)^2$):

$$\begin{aligned} g_z - (\varepsilon + m_e - \bar{V})f &= 0, \\ f_z + (\varepsilon - m_e - \bar{V})g &= 0, \end{aligned} \quad (30)$$

where $g_z \equiv dg/dz$, $f_z \equiv df/dz$; ε is the energy eigenvalue of the Dirac equation; the bispinor $\psi_e = \begin{pmatrix} \varphi_e \\ \chi_e \end{pmatrix}$ of the electron is decomposed into $\varphi_e = \begin{pmatrix} 0 \\ g(z) \exp(-\rho^2/4a_H^2) \end{pmatrix}$, $\chi_e = \begin{pmatrix} 0 \\ if(z) \exp(-\rho^2/4a_H^2) \end{pmatrix}$. The averaged potential energy $\bar{V}(z)$ is defined in the same way as it was done in Sec. III.

The analytical formula which describes the dependence of the atomic energies ε on B was derived in [5]:

$$Ze^2 \ln \left(2 \frac{\sqrt{m_e^2 - \varepsilon^2}}{\sqrt{eB}} \right) + \arctan \left(\sqrt{\frac{m_e + \varepsilon}{m_e - \varepsilon}} \right) + \arg \Gamma \left(-\frac{Ze^2 \varepsilon}{\sqrt{m_e^2 - \varepsilon^2}} + iZe^2 \right) - \arg \Gamma(1 + 2iZe^2) - \frac{Ze^2}{2}(\ln 2 + \gamma) = \frac{\pi}{2} + n\pi, \quad (31)$$

where the argument of the gamma function is given by

$$\arg \Gamma(x + iy) = -\gamma y + \sum_{k=1}^{\infty} \left(\frac{y}{k} - \arctan \frac{y}{x + k - 1} \right). \quad (32)$$

For the ground level at $\varepsilon > 0$ one should take $n = 0$, while for $\varepsilon < 0$ it should be changed to $n = -1$.

Substituting $\varepsilon = -m_e$ into (31) the formula for the critical nucleus charge in a magnetic field was found in [5]:

$$\frac{B}{B_0} = 2(Z_{cr}e^2)^2 \exp \left(-\gamma + \frac{\pi - 2 \arg \Gamma(1 + 2iZ_{cr}e^2)}{Z_{cr}e^2} \right). \quad (33)$$

According to this formula the critical nucleus charge diminishes with the magnetic field and for $B \approx 10^2 B_0$ the uranium becomes critical (equations (31) and (33) are valid for $B > \max \{ (Ze^2)^2 B_0, B_0 / (Ze^2)^2 \}$).

To satisfy the matching condition used to derive the formula (31) the potential should be Coulomb at distances $l > z_0$, where $z_0 \ll Ze^2/(2m_e)$ is the matching point (see [5] for details). This condition is violated for the screened potential. Since we do not have an analytical formula for the energy levels which takes screening into account in the relativistic case, in [7] we solved the Dirac equation numerically. In order to do this we followed the paper [5] where the system (30) was transformed into one second order differential equation for $g(z)$. By substituting $g(z) = (\varepsilon + m_e - \bar{V})^{1/2} \chi(z)$ a Schrödinger-like equation for the function $\chi(z)$ was obtained in [5]:⁵

$$\frac{d^2 \chi}{dz^2} + 2m_e(E - U)\chi = 0, \quad (34)$$

$$E = \frac{\varepsilon^2 - m_e^2}{2m_e}, \quad U = \frac{\varepsilon}{m_e} \bar{V} - \frac{1}{2m_e} \bar{V}^2 + \frac{\bar{V}''}{4m_e(\varepsilon + m_e - \bar{V})} + \frac{3/8(\bar{V}')^2}{m_e(\varepsilon + m_e - \bar{V})^2}.$$

⁵ This method of reduction of the Dirac equation to a Schrödinger-like equation was originally proposed by V.S. Popov to analyze the critical nucleus charge phenomenon qualitatively.

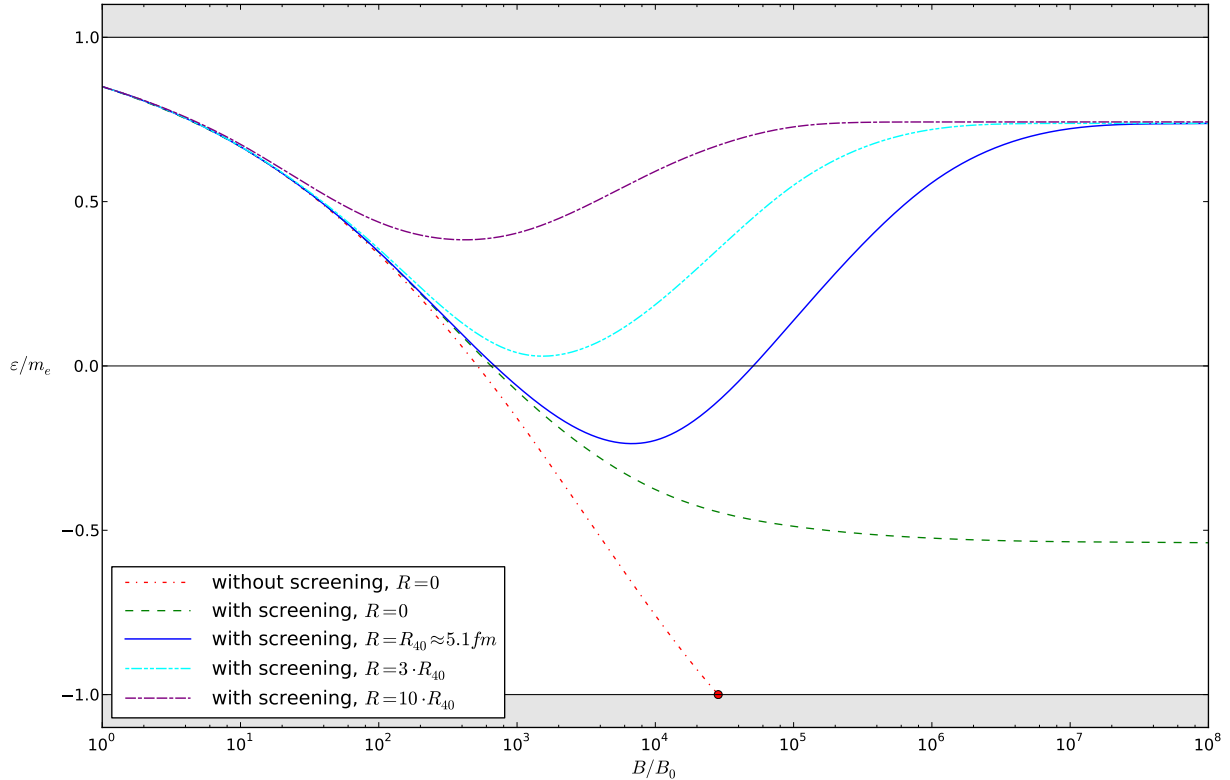


FIG. 2. The dependence of the ground energy level on the magnetic field for $Z = 40$. The dot-dashed (red) line corresponds to a pointlike potential without screening, the dashed (green) line — to a pointlike potential with screening, the solid (blue) line — to a potential which takes into account both screening and the finite size of the nucleus $R = R_{40} \approx 5.1$ fm. Cyan and purple lines (linestyles are shown in legend) corresponds to (hypothetical) larger nucleus radii, $R = 3R_{40}$ and $R = 10R_{40}$.

In [7] we used the equation (34) for numerical calculations. The numerical results for the pointlike nucleus were presented in [7] where the freezing of the ground energy level was obtained in the relativistic domain and the values of the critical magnetic fields were found for $Z \geq 50$. Nuclei with $Z < 50$ do not become critical due to screening.

Our next step is to generalize the results of [7] in order to take the finite nucleus size into account. Let us note that in most cases the averaged potential energy \bar{V} is smooth even when the potential $\Phi(\rho, z)$ has a cusp at the nucleus boundary. However due to our approximations in averaging $\bar{V}^{(1)}(z)$ done in the Sec. III the potential does have this cusp which is quite significant for $B \gg 1/(eR^2)$. It means that we have a δ -singularity in the

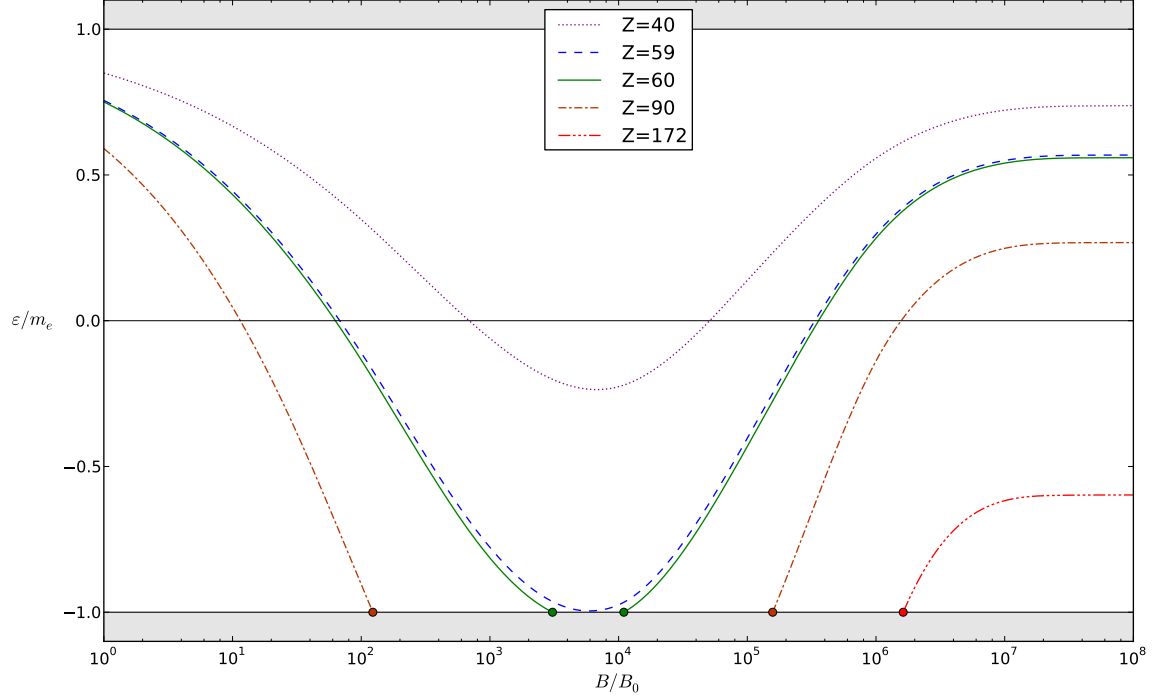


FIG. 3. The dependence of the ground energy level on magnetic field for $Z = 40, 59, 60, 90, 172$. The correspondence between charge Z and color (linestyle) is shown in the legend.

effective potential U and this should be taken into account in numerical calculations.

Substituting $\bar{V}^{(1)}(z)$ for $\bar{V}(z)$ we checked that the equation (34) gives the values of the binding energy for hydrogen $E = \frac{\varepsilon^2 - m_e^2}{2m_e} \equiv -\frac{m_e e^4}{2} \lambda^2$, which is very close to the ones obtained from the Schrödinger equation.

Substituting the nucleus radius $R_Z = r_0 A^{1/3}$ and $\bar{V}(z) = Z\bar{V}^{(1)}(z)$ into (34) we numerically calculate the value of the ground state energy of the hydrogen-like ion with charge Z .⁶ In Fig. 2 the dependence of the ground state energy on B is shown for $Z = 40$. We observe the rising of the ground energy level in the relativistic domain. The curves for a nucleus with $Z = 40$ and radii $3R_{40}$ and $10R_{40}$ are plotted to check how the energy depends on nucleus radius. We see that the rising starts at $B \sim 1/(eR^2)$ and stops at $B \sim 1/(e^3 R^2)$. The limiting energy does not depend on R .

In Fig. 3 the dependence of the ground state energy on the magnetic field for $Z = 40, 59, 60, 90, 172$ is shown. We see that ions with $Z < 60$ never become critical while a nucleus with $Z = 60$ is critical only within the small range of magnetic fields around

⁶ In the numerical calculations we are using $r_0 = 1.1 fm$ and $A = 2.5Z$.

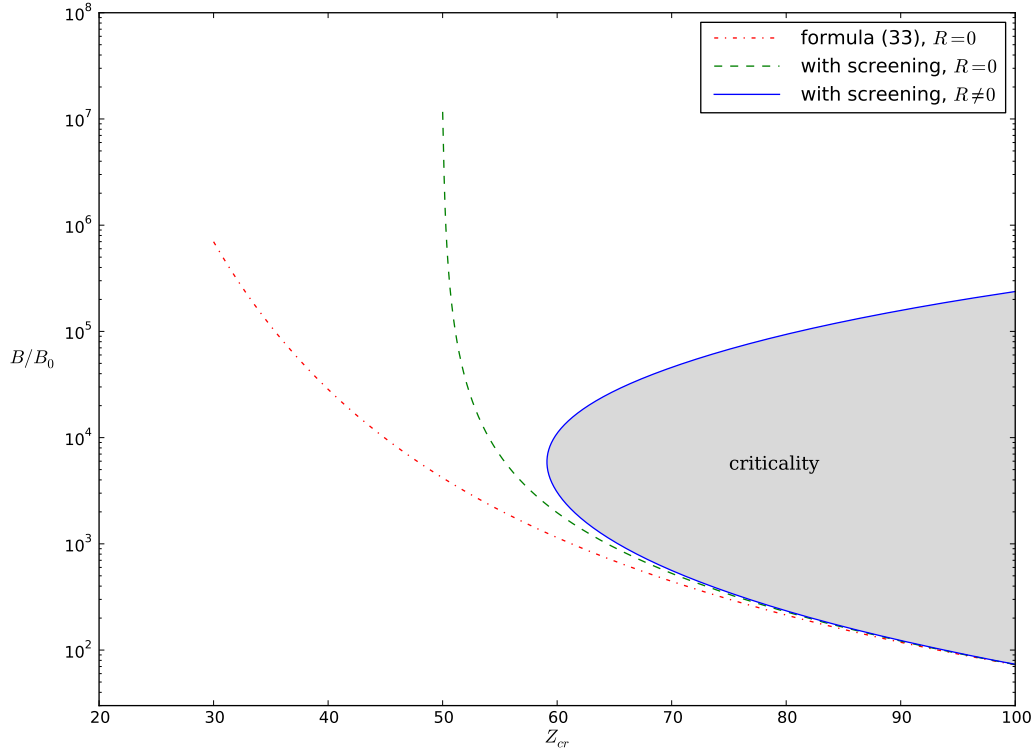


FIG. 4. The values of magnetic fields at which the nuclei with charge Z_{cr} becomes critical: a) without screening according to eq. (33), dot-dashed (red) line; b) numerical results with screening for pointlike nucleus, dashed (green) line; c) numerical results which take finite size of the nucleus into account, solid (blue) line.

$B \approx 10^4 B_0$. For larger Z the range of magnetic fields in which ions are critical becomes wider. Ions become critical at B a little bit larger than the critical field for a pointlike nucleus. The rising of the ground energy level makes the ions noncritical for strong enough magnetic fields. Even an ion with $Z = 172$ becomes noncritical for $B > 1.6 \cdot 10^6 B_0$ while it is critical for $B = 0$. To estimate the value of the nucleus charge at which the “final”, or “second”, freezing energy reaches the lower continuum we demanded that the nucleus should be critical for $B = 10^8 B_0$ and found that it is satisfied for $Z \geq 210$. It means that only ions with $Z \geq 210$ remains critical regardless of the value of the magnetic fields.

In Fig. 4 the dependence of critical nucleus charge on B is shown.

V. THE HYDROGEN ATOMIC LEVELS IN A SUPERSTRONG B AND THE PROTON MOTION

The two-body problem in the presence of a homogeneous magnetic field constant in time was analyzed in the papers [8–10] where it was found that for an electrically neutral system the separation of the center-of-mass and relative coordinates can be carried out explicitly. According to Eq. (12) from [10] the Hamiltonian which describes the relative motion of the electron and the proton in an external magnetic field \vec{B} looks like:

$$\hat{H}_{rel} = \frac{\vec{K}^2}{2M} + \frac{e}{M} (\vec{K} \times \vec{B}) \cdot \vec{r} + \frac{1}{2m_r} \vec{p}^2 + \frac{e}{2} \left(\frac{1}{m_e} - \frac{1}{m_p} \right) \vec{B} \cdot (\vec{r} \times \vec{p}) + \frac{e^2}{8m_r} (\vec{B} \times \vec{r})^2 + V(r), \quad (35)$$

where the momentum \vec{K} is an eigenvalue of the generalized momentum operator responsible for the motion of the center-of-mass of the atom, $\vec{r} \equiv \vec{r}_e - \vec{r}_p$ and $\vec{p} \equiv -i\partial/\partial\vec{r}$ are the relative coordinate and momentum of the electron and proton, $M = m_e + m_p$, $m_r \equiv m_e m_p / M$ is the reduced mass, and $V(r)$ describes the Coulomb interaction between the electron and the proton. The screening of the Coulomb interaction in a superstrong \vec{B} should be taken into account as well [1, 2].

Let us consider an atom at rest ($\vec{K} = 0$) in a strong magnetic field $B \gg m_e^2 e^3$ in which the adiabatic approximation is applicable. The relative motion of the electron and proton in the direction of the magnetic field is determined by the potential $V(z)$, while in the direction transverse to the magnetic field it is determined by \vec{B} .

Neglecting in a first approximation the Coulomb attraction for the atomic energy levels from (35) we obtain:

$$E_{n_\rho m} = \frac{eB}{m_e} \left(n_\rho + \frac{|m| + m + 1}{2} \right) + \frac{eB}{m_p} \left(n_\rho + \frac{|m| - m + 1}{2} \right). \quad (36)$$

In what follows we will be interested in the states of the hydrogen atom which originate from the lowest Landau level (LLL), for which $n_\rho = 0$, $m = 0, -1, -2, \dots$ and the electron spin is antiparallel to \vec{B} . For LLL the contribution of the first term in (36) summed with the energy of interaction of the electron spin with the external magnetic field \vec{B} is zero while the second term gives a nonzero contribution. Thus the energies of the atomic states with different m are shifted by

$$\Delta E_m = \frac{eB}{m_p} |m|. \quad (37)$$

Taking into account the motion along the z -axis governed by the (screened) Coulomb potential we obtain the following expression for the atomic energies:

$$E = -\frac{m_r e^4}{2} \lambda^2 + \frac{eB|m|}{m_p} \equiv E_\lambda + \frac{eB|m|}{m_p}, \quad (38)$$

where in the first term m_e can be safely substituted for the reduced mass m_r since the numerical difference between m_r and m_e is very small. The values of λ are determined by the following transcendental equation (see [2], Eq. (57)):

$$\ln \left(\frac{H}{1 + \frac{e^6}{3\pi} H} \right) = \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|), \quad (39)$$

where $H \equiv B/(m_e^2 e^3)$ is the magnetic field in units of atomic magnetic field, ψ is the logarithmic derivation of the gamma function and $\gamma = 0.5772 \dots$ is the Euler's constant.

For each $m = 0, -1, -2, \dots$ the solution of (39) produces a tower of states; the tower with the lowest values of E_λ corresponds to $m = 0$. The values of E_λ in the limit $B \gg 3\pi m_e^2/e^3$ (or $H \gg 3\pi/e^6$) are shown in [2], Fig. 10. However the atomic energies of the states from different towers are shifted by the value $\Delta E_m = eB|m|/m_p$ and since ΔE_m grows linearly with the magnetic field, for strong enough B this shift is big.

Let us consider $B = 2 \cdot 10^3 m_e^2 e^3 \approx 4.7 \cdot 10^{12}$ Gauss which was used in the calculation of the energies in the Table 1 of [10]. For $m = 0$ for the ground level from (39) we obtain $\lambda_0^0 = 4.3$, $E_\lambda(m = 0) = -255 \text{ eV}$. For $m = -1$ the ground level according to (39) corresponds to $\lambda_0^{-1} = 3.8$, $E_\lambda(m = -1) = -193 \text{ eV}$ which is 62 eV above $E_\lambda(m = 0)$. The proton motion increase the atomic energies of the $m = -1$ tower by $eB/m_p = 30 \text{ eV}$ and the difference of energies with account of the finite proton mass becomes 92 eV (according to Table 1 of [10] it equals 93.3 eV). For stronger B the second term in (38) starts to dominate over the first one.

VI. SPIN-SPIN INTERACTION IN HYDROGEN, HEAVY IONS AND POSITRONIUM IN A STRONG EXTERNAL MAGNETIC FIELD

The interaction between the proton and electron spins leads to the hyperfine splitting of the hydrogen atomic levels. Being proportional to $\mu_e \mu_p |\psi(0)|^2 \sim m_e \alpha^4 (m_e/m_p)$ it is much smaller than the atomic energies, which are of the order of $\alpha^2 m_e$. In the case of a strong external magnetic field $B \gg m_e^2 e^3$ the spin-spin interaction considerably grows: $|\psi(0)|_B^2 \sim$

$|\psi(0)|^2 \left(\frac{a_B}{a_H}\right)^2 = |\psi(0)|^2 \left(\frac{B}{m_e^2 e^3}\right)$, leading to a linear increase of the hyperfine splitting with magnetic field⁷:

$$E_{SS} \sim m_e \alpha^2 \frac{m_e}{m_p} \frac{B}{B_0}. \quad (40)$$

It follows that at $B \sim 10^5 B_0$ the spin-spin interaction energy becomes of the order of the freezing energy of the hydrogen ground level $E_0 \approx -1.7 \text{ keV}$, and if a linear growth of E_{SS} with B would take place for $B > 10^5 B_0$ it would determine the value of the ground state atomic energy E_0 . However just at $B \sim 10^5 B_0$ the Landau radius a_H approaches the proton charge radius and a power formfactor suppression of E_{SS} occurs preventing it from growing further.

The energy of the spin-spin interaction in heavy hydrogenlike ions is enhanced by a factor Z which originates from the $1/a_B = Z m_e \alpha$ factor in the expression for $|\psi(0)|^2$. Since protons and neutrons from completely filled nuclei shells do not contribute to the magnetic moment of the nucleus, for $B \lesssim 10^5 B_0$ the extra term in the energy is considerably smaller than the value of the electron mass and the consideration of nuclei criticality in strong B does not change substantially. (One should also take into account that the formfactor suppression in heavy ions starts at smaller B).

It would be very interesting to understand to which shift of the energy of the positronium ground state in a superstrong magnetic field the spin-spin interaction of the electron and positron leads. In the absence of external magnetic field, due to this interaction, the ground state of the parapositronium is lighter than the ground state of the ortopositronium [13, 14]:

$$E(^3S_1) - E(^1S_0) = \frac{7}{12} \alpha^2 m_e e^4 = \frac{7}{12} e^8 m_e. \quad (41)$$

The behaviour of the positronium energy levels in an external magnetic field has several specific features [14], and in fields $B \gtrsim e^8 m_e^2 / e$ the state with lower energy is a mixture of ortopositronium and parapositronium ground states in which the electron spin is oriented in the direction opposite to the magnetic field, while the spin of the positron is directed along B . Its energy shift due to the spin-spin interaction is of the order of:

$$\Delta E \sim \mu_e^2 |\psi(0)|^2 \sim m_e e^4 \frac{B}{B_0}, \quad (42)$$

⁷ M. A. Andreichikov, B. O. Kerbikov, private communication

and for $B > 10^4 B_0$ the ground state positronium energy could become lower than $-2m_e$ which should lead to the production of e^+e^- pairs from the vacuum (see [15] as well).

However the spin-spin interaction Hamiltonian is determined as a nonrelativistic expansion of the e^+e^- scattering amplitude, the expansion parameter being p^2/m_e^2 [13]. In the case of a strong external magnetic field $p^2/m_e^2 \sim 1/(a_H^2 m_e^2) = B/B_0$, and for $B \gtrsim B_0$ the correctness of the formulae (40) and (42) is doubtful.

Let us remind that the anomalous electron magnetic moment leads to a linear growing with B of correction to the lowest Landau level energy. However this behaviour is valid only for $B \lesssim B_0$, while for $B \gtrsim B_0$ the strong linear dependence on B is replaced by a weak double logarithmic one ([16], see also [2]).

Concluding this section let us state that the behaviour of the spin-spin interaction in atoms and positronium at strong external magnetic fields $B \gtrsim B_0$ deserves further study.

VII. CONCLUSIONS

In a magnetic field $B \gg m_e^2/e^3$ the potential of a pointlike charge becomes screened due to large radiative corrections. The screened potential has a Coulomb behaviour along the z axis at distances $a_H < |z| < 1/\sqrt{e^3 B}$ and $|z| \gtrsim 1/m_e$ and these regions define the ground state atomic energy. Distributing the pointlike charge within a domain of the size of the nuclear radius R leads to a less singular behaviour of the potential at distances $|z| < R$. When $e^3 B$ approaches $1/R^2$ the ground state energy approaches the limiting value. For hydrogen the limiting value of λ^{gr} which defines the ground state energy $E_0 = -(m_e e^4/2)(\lambda^{\text{gr}})^2$ is equal to 6.9 instead of 11.2 obtained in [2] for a pointlike charge, and $E_0^{\text{lim}} = -0.65 \text{ keV}$ instead of -1.7 keV .

The same phenomenon of going up of the ground energy level is obtained numerically in the relativistic domain for hydrogen-like ions. It leads to a nontrivial dependence of the critical nucleus charge on the magnetic field: the nuclei with $Z < 60$ never become critical while the ions with $60 \leq Z < 210$ are critical only within a finite range of magnetic fields. At $Z = 210$ a “second” freezing energy reaches the lower continuum and the nuclei with $Z \geq 210$ are critical at any B .

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- [1] A.E. Shabad, V.V. Usov, Phys. Rev. Lett. **98** (2007) 180403;
A.E. Shabad, V.V. Usov, Phys. Rev. D **77** (2008) 025001.
 - [2] B. Machet, M.I. Vysotsky, Phys. Rev. D **83** (2011) 025022.
 - [3] J. Beringer et al. (Particle Data Group), Phys. Rev. D **86** (2012) 010001.
 - [4] W. Pieper, W. Greiner, Z. Physik **218** (1969) 327;
V.S. Popov, Pis'ma Zh. Eksp. Teor. Fiz. **11** (1970) 254;
V.S. Popov, Zh. Eksp. Teor. Fiz. **59** (1970) 965;
Ya.B. Zeldovich, V.S. Popov, UFN **105** (1971) 403;
V.S. Popov, Yad. Fiz. **14** (1971) 458.
 - [5] V.N. Oraevskii, A.I. Rez, V.B. Semikoz, Zh. Eksp. Teor. Fiz. **72** (1977) 820 [Sov. Phys. JETP **45** (1977) 428].
 - [6] P. Schlüter, G. Soff, K.-H. Wietschorke, W. Greiner, J. Phys. B: At. Mol. Phys. **18** (1985) 1685.
 - [7] S. I. Godunov, B. Machet, M.I. Vysotsky, Phys. Rev. D **85** (2012) 044058.
 - [8] L. P. Gor'kov, I. E. Dzyaloshinskii, Zh. Eksp. Teor. Fiz. **53** (1968) 717 [J. Exp. Theor. Phys. **26** (1968) 449].
 - [9] J. E. Avron, I. W. Herbst, B. Simon, Phys. Rev. A **20** (1979) 2287.
 - [10] H. Herold, H. Ruder, G. Wunner, J. Phys. B: At. Mol. Phys. **14** (1981) 751.
 - [11] M. A. Andreichikov, B. O. Kerbikov, Yu. A. Simonov, arXiv:1304.2516 (2013).
 - [12] B.M. Karnakov, V.S. Popov, Zh. Eksp. Teor. Fiz. **124** (2003) 996 [J. Exp. Theor. Phys. **97** (2003) 890];
B.M. Karnakov, V.S. Popov, Zh. Eksp. Teor. Fiz. **141** (2012) 5. [J. Exp. Theor. Phys. **114** (2012) 1]

- [13] V.B. Berestetskiy, L.D. Landau, Zh. Eksp. Teor. Fiz. **19** (1949) 673.
- [14] V.B. Berestetskiy, Zh. Eksp. Teor. Fiz. **19** (1949) 1130.
- [15] A.E. Shabad, V.V. Usov, Phys. Rev. Lett. **96** (2006) 180401;
- [16] I.M. Ternov, V.G. Bagrov, V.A. Bordovitsyn, and O.F. Dorofeev, Zh. Eksp. Teor. Fiz. **55**, 2273 (1968) [Sov. Phys. JETP **28**, 1206 (1969)]; B. Jankovichi, Phys. Rev. **187**, 2275 (1969).